Compactness Theorem

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Abstract

Marker [2] elaborated a proof of Henkin's Construction for the Compactness Theorem. Halvorson [1] also provided a very *friendly* proof of the Henkin Construction of the Compactness Theorem. This proof presented in this write-up is primarily based on Wim Veldman's proof, with the gaps filled with the former two proofs. Note, this is a constructive proof of the Compactness Theorem, and therefore using Zorn's Lemma is not allowed. I would want to thank Kaspar Hagens from Radboud University, Nijmegen, for his helpful insights.

1 Henkin's Construction

Theorem 1.1. *Let Γ be a theory in $\mathcal{L}$. If every finite subset of Γ has a model, then Γ itself has a finite or countable model.*

Proof. The language $\mathcal{L}$ contains relation symbols $\mathsf{R}_0, \ldots, \mathsf{R}_{m-1}, =$, and function symbols $\mathsf{F}_0, \ldots, \mathsf{F}_{\mathsf{n}-1}$, and individual constants $\mathsf{c}_0, \ldots, \mathsf{c}_{\mathsf{p}-1}$. Let $\mathcal{L}'$ be the language obtained by adding to $\mathcal{L}$ a countable sequence $\mathsf{a}_0, \mathsf{a}_1, \mathsf{a}_2, \ldots$ of *new* individual constants (please note that the constants are being added to the language, not the structure!). Now let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an enumeration of *all* (not only the true) sentences from the extended language $\mathcal{L}'$. We now define an increasing sequence $\Delta_0, \Delta_1, \ldots$ of finite sets of sentences from $\mathcal{L}'$ via induction : $\Delta_0 := \emptyset$ and for each $n \in \mathbb{N}$,

Δ_{n+1}	$:=$	Δ_n	if there exists a finite subset B of Γ such that $B \cup \Delta_n \cup \{\varphi_n\}$ has no model.
Δ_{n+1}	$:=$	$\Delta_n \cup \{\varphi_n\}$	if for every finite subset B of Γ the set $B \cup \Delta_n \cup \{\varphi_n\}$ has a model and the formula φ_n is not an existential formula, that is, it does not have the form $\exists x[\psi]$
Δ_{n+1}	$:=$	$\Delta_n \cup \{\varphi_n, S^x_{\mathsf{a}_m}\psi\}$	if φ_n has the form $\exists x[\psi]$ and for every finite subset B of Γ the set $B \cup \Delta_n \cup \{\varphi_n\}$ has a model and m is the least number k such that the individual constant a_k does not occur in φ_n and not in any formula from Δ_n.

Finally, we define $\Delta := \bigcup_{n \in \mathbb{N}} \Delta_n$. We now observe that that Δ has the following properties :

(i) For every n, for every finite subset B of Γ, the set $B \cup \Delta_n$ has a model. We prove this by induction : for $n = 0$, we know $B \cup \Delta_0 = B$ has a model (as every finite subset of Γ has a model). Now, by induction hypothesis, assume $B \cup \Delta_{n-1}$ has a model. If we have the first case of the definition of Δ, then $\Delta_{n-1} = \Delta_n$ hence $B \cup \Delta_n$ has a model. If, instead, the second case applies, then $B \cup \Delta_{n-1} \cup \{\varphi_{n-1}\}$ has a model. Since $\Delta_n = \Delta_{n-1} \cup \{\varphi_{n-1}\}$,

$B \cup \Delta_n$ has a model. For the third case, we have that $B \cup \Delta_{n-1} \cup \{\varphi_{n-1}\}$ has a model (say $\mathfrak{A}$) and m is the least number k such that the individual constant a_k does not occur in φ_{n-1} and not in any formula from Δ_{n-1}. φ_{n-1} is of the form $\exists x[\psi]$, therefore, by the definition of $L-$structure, there exists a b in $B \cup \Delta_{n-1} \cup \{\varphi_{n-1}\}$ such that $\mathfrak{A} \vDash \phi(b)$. Now we shall expand the structure $\mathfrak{A}$ with the constant a_k (which was added to $\mathcal{L}$), having as *interpretation* b. Therefore, $B \cup \Delta_{n-1} \cup \{\varphi_{n-1}, S^x_{\mathsf{a}_k}\psi\}$ has as model $(\mathfrak{A}, ..., b)$. Substituting $\Delta_n = \Delta_{n-1} \cup \{\varphi_{n-1}, S^x_{\mathsf{a}_k}\psi\}$ gives us that $B \cup \Delta_n$ has a model.

(ii) $\Gamma \subseteq \Delta$. Let $\varphi \in \Gamma$. Determine m such that $\varphi = \varphi_m$. Since $\varphi_m \in \Gamma$, therefore, for any finite subset B of Γ, $B \cup \{\varphi_m\}$ is a finite subset of Γ, therefore, by (i), $B \cup \Delta_m \cup \{\varphi_m\}$ has a model. By the construction of Δ, $\varphi_m = \varphi \in \Delta$.

(iii) For every n, either the sentence φ_n belongs to Δ or the sentence $\neg\varphi_n$ belongs to n. Suppose φ_n does not belong to Δ, therefore, we can determine a finite subset B_0 of Δ such that $B_0 \cup \Delta_n \cup \{\varphi_n\}$ has no model. We then determine m such that $\varphi_m = \neg\varphi_n$. We claim that for every finite subset B' of Γ, the set $B' \cup \Delta_m \cup \{\varphi_m\}$ has a model. Indeed, $B_0 \cup B' \cup \Delta_m \cup \Delta_n$ has a model (say $\mathfrak{A}$) via (i), because either $\Delta_n \subseteq \Delta_m$ or $\Delta_m \subseteq \Delta_n$ and $B_0 \cup B'$ is a finite subset of Γ. Now, since $B_0 \cup \Delta_n \cup \{\varphi_n\}$ has no model, therefore $\mathfrak{A} \nvDash \varphi_n$. By the definition of a model, $\mathfrak{A} \vDash \neg\varphi_n$. Therefore, $B' \cup \Delta_m \cup \{\varphi_m\}$ has as model $\mathfrak{A}$. By the construction of Δ, $\varphi_m \in \Delta_{m+1} \subset \Delta$.

(iv) For all sentences ψ in $\mathcal{L}'$, $\neg(\psi)$ belongs to Δ if and only if ψ does not belong to Δ.

($\Rightarrow$) Say $\neg\varphi \in \Delta$, let $\varphi_m = \neg\varphi$ and let $\varphi_n = \varphi$. Since φ_m belongs to Δ, by the construction of Δ, for any subset B_0 of Γ, $B_0 \cup \Delta_m \cup \{\varphi_m\} = B_0 \cup \Delta_{m+1}$ must have a model, and contains φ_m. By (i), for any finite subset B_1 of Γ, $B_0 \cup B_1 \cup \Delta_{m+1} \cup \Delta_{n+1}$ has a model and contains φ_m. Hence, $B_0 \cup B_1 \cup \Delta_{m+1} \cup \Delta_{n+1}$ does not contain φ_n. Since n is the only stage at which φ_n could have been added, $\varphi_n = \varphi$ does not belong to Δ.

($\Leftarrow$) (iii).

(v) For all sentences φ, ψ in $\mathcal{L}'$, $(\varphi) \wedge (\psi)$ belongs to Δ if and only if φ and ψ both belong to Δ.

($\Rightarrow$) We have that $(\varphi) \wedge (\psi)$ belong to Δ. Determine l such that $\varphi_l = (\varphi) \wedge (\psi)$, and determine m, n such that $\varphi_m = \varphi$ and $\varphi_n = \psi$. Since $\varphi_l = (\varphi) \wedge (\psi)$ belongs to Δ, for any finite subset B_0 of Γ, $B_0 \cup \Delta_l \cup \{\varphi_l\}$ has a model (by construction of Δ) and is equal to $B_0 \cup \Delta_{l+1}$. Now, by (i), for any subset B_1 of Γ, $B_0 \cup B_1 \cup \Delta_{l+1} \cup \Delta_{m+1}$ has a model. Since $B_0 \cup B_1 \cup \Delta_{l+1} \cup \Delta_{m+1}$ has a model and contains φ_l, we know that $\neg\varphi_m \notin B_0 \cup B_1 \cup \Delta_{l+1} \cup \Delta_m$ and hence $\neg\varphi_m$ is not in Δ (as it is only at stage m that it is added). Hence, by (iv), we have φ_m belongs to Δ. Similar method applies to n.

($\Leftarrow$) We have that φ and ψ are in Δ, hence, by construction of Δ, for finite subsets B_0 and B_1 of Γ, $B_0 \cup \Delta_m \cup \{\varphi\}$ and $B_1 \cup \Delta_n \cup \{\psi\}$ have models and hence $B_0 \cup \Delta_{m+1}$ and $B_1 \cup \Delta_{n+1}$ have models, and each respectively contain φ and ψ. Determine l such that $\varphi_l = (\varphi) \wedge (\psi)$. By (i), we have that for any finite subset B_2 of Γ, $\Gamma^* := B_0 \cup B_1 \cup B_2 \cup \Delta_{m+1} \cup \Delta_{n+1} \cup \Delta_{l+1}$ has a model (say $\mathfrak{A}$). Since $\varphi \in \Gamma^*$ and $\psi \in \Gamma^*$, $\mathfrak{A} \vDash \varphi$ and $\mathfrak{A} \vDash \psi$, hence $\mathfrak{A} \vDash (\varphi) \wedge (\psi)$. Hence, by the construction of Δ, $(\varphi) \wedge (\psi) \in \Delta_{l+1} \subset \Delta$.

(vi) For any closed term t in $\mathbf{Term}(\mathcal{L}')$, $t = t$ belongs to Δ. Let $\varphi_m = (t \neq t)$. At stage m, we verify for any finite subset B of Γ, if $B \cup \Delta_m \cup \{\varphi_m\}$ has a model. Since $(t \neq t)$ has no model, therefore, $(t \neq t)$ does not belong to Δ. Hence, by (iii), $(t = t)$ belongs to Δ.

(vii) For any two closed terms t, t' and a term $\phi(x)$ of $\mathbf{Term}(\mathcal{L}')$, if $t = t'$ and $\phi(t)$ belongs to Δ, then $\phi(t')$ belongs to Δ : determine $\varphi_m = (t = t')$ and $\varphi_n = \phi(x)$. For stage m, by construction of Δ, for any finite subset B_0 of Γ, $B_0 \cup \Delta_m \cup \{\varphi_m\} = B_0 \cup \Delta_{m+1}$ has a model. Similarly for stage n, we have that for any finite subset B_1 of Γ, $B_1 \cup \Delta_n \cup \{\varphi_n\} = B_1 \cup \Delta_{n+1}$ has a model. By (i), $B_0 \cup B_1 \cup \Delta_{m+1} \cup \Delta_{n+1}$ has a model. Now, determine l such that $\varphi_m = \neg\phi(t')$. Now, consider, for any finite subset B_2 of Γ, we know via (i) that $\Gamma^* := B_0 \cup B_1 \cup B_2 \cup \Delta_{m+1} \cup \Delta_{n+1} \cup \Delta_{l+1}$ has a model, and it contains $(t = t')$ and $\phi(t)$. We observe that $\Gamma^* \cup \{\neg\phi(t')\}$ has no model, therefore, since l is the only stage that it $\varphi_l = \neg\phi(t')$ could have been added, $\neg\phi(t)$ does not belong to Δ. By (iii), $\phi(t')$ belongs to Δ.

(viii) For all sentences φ of the form $\exists x[\psi(x)]$: the sentence $\exists x[\psi(x)]$ belongs to Δ if and only if there exists individual constant a_i such that the sentence $S^x_{\mathsf{a}_i}$ belongs to Δ. This directly follows from the construction of Δ (from its third case).

(ix) For any formula $\varphi(x)$ and for any constant c in $\mathcal{L}$, if $\phi(c)$ belongs to Δ, then the sentence $\exists x\phi(x)$ belongs to Δ : determine m such that $\varphi_m = \neg\exists x\phi(x)$, and determine $\varphi_n = \phi(c)$. Since $\varphi_n = \phi(c)$ belongs to Δ, by construction of Δ, for any finite subset B_0 of Γ, we know $B_0 \cup \Delta_n \cup \{\phi(c)\} = B_0 \cup \Delta_{n+1}$ has a model. At stage m, we observe, by (i), that for any finite subset B_1 of Γ, we have $\Gamma^* := B_0 \cup B_1 \cup \Delta_{n+1} \cup \Delta_{m+1}$ has a model, say $\mathfrak{A}$. We also observe that since $\phi(c)$ belongs to Γ^*, we have that $\Gamma^* \cup \{\neg\exists x[\phi(x)]\}$ has no model, therefore, $\neg\exists x[\phi(x)]$ does not belong to Δ as it could only be added at stage m. Therefore, by (iii), $\exists x[\phi(x)]$ belongs to Δ.

We construct $\mathcal{M} = (A, R_0^{\mathcal{M}}, \ldots, R_{m-1}^{\mathcal{M}}, f_0^{\mathcal{M}}, \ldots, f_{n-1}^{\mathcal{M}}, c_0, \ldots, c_{p-1}^{\mathcal{M}}, a_0^{\mathcal{M}}, a_1^{\mathcal{M}}, \ldots)$ realising Δ.

We first build the domain of the structure $\mathcal{M}$. Consider the set $\mathbf{Term}(\mathcal{L}')$ of all *closed* (individual) terms of the extended language $\mathcal{L}'$ (an individual term is called *closed* if it does not contain any individual variable). We define a binary relation, $\sim_\Delta$ on this set as follows :

$$\text{for all closed terms } s, t : s \sim_\Delta t := \text{ the sentence } s = t \text{ belongs to } \Delta$$

We now demonstrate that $\sim_\Delta$ is an equivalence relation on $\mathbf{Term}(\mathcal{L}')$:

(i) *Reflexive* : directly follows from (vi).

(ii) *Symmetry* : Take $\phi(t) = (t = c)$. By (vi), we have that $\phi(c) = (c = c)$ belongs to Δ. If $(c = d)$ in Δ, by (vii), $\phi(d) = (d = c)$ belongs to Δ.

(iii) *Transitive* : We have $(a = b)$ and $(b = c)$ in Δ. Take $\phi(t) = (a = t)$. By (vi), we have that $\phi(a) = (a = a)$ belongs to Δ. We also have $\phi(b) = (a = b)$ belongs to Δ. Since we have $(b = c)$ in Δ, by (vii), we know that $\phi(c) = (a = c)$ belongs to Δ.

We define $A := \{[s] : s \in \mathbf{Term}(\mathcal{L}')\}$. We also note an equivalence class always induces a partition on the underlying set (in this case, Δ).

We define the relations of the structure $\mathcal{M}$. For every $i < m$, for every sequence $(s_0, s_1, \ldots, s_{k_i-1})$ of elements of $\mathbf{Term}(\mathcal{L}')$, we define :

$$\langle [s_0], [s_1], \ldots, [s_{k_i-1}] \rangle \text{ belongs to } R_i^{\mathcal{M}} \text{ if and only if } \mathsf{R}_i(s_0, s_1, \ldots, s_{k_i-1}) \text{ belongs to } \Delta$$

This definition is unambiguous, as for all sequences $(s_0, s_1, \ldots, s_{k_i-1})$ and $(t_0, t_1, \ldots, t_{k_i-1})$ of elements of $\mathbf{Term}(\mathcal{L}')$, if for each $j < k_i$, the sentence $s_j = t_j$ belongs to Δ, then the sentence $\mathsf{R}_i(s_0, s_1, \ldots, s_{k_i-1})$ belongs to Δ if and only if the sentence $\mathsf{R}_i(t_0, t_1, \ldots, t_{k_i-1})$ in Δ.

Next, we define the functions of the structure $\mathcal{M}$. For every $j < n$, for every sequence $(s_0, s_1, \ldots, s_{l_j-1})$ of elements of $\mathbf{Term}(\mathcal{L}')$, we define :

$$f_j^{\mathcal{M}}([s_0], [s_1], \ldots, [s_{k_i-1}]) := [\mathsf{F}_j(s_0, s_1, \ldots, s_{k_i-1})]$$

This definition is unambiguous, as for all sequences $(s_0, s_1, \ldots, s_{k_i-1})$ and $(t_0, t_1, \ldots, t_{k_i-1})$ of elements of $\mathbf{Term}(\mathcal{L}')$, if for every $i < l_j$, the sentence $s_i = t_i$ belongs to Δ, then the sentence $\mathsf{F}_i(s_0, s_1, \ldots, s_{l_j-1}) = \mathsf{F}_j(t_0, t_1, \ldots, t_{l_j-1})$ belongs to Δ. Hence, this definition is equivalent to $f_j^{\mathcal{M}}([s_0], [s_1], \ldots, [s_{k_i-1}]) = [t]$ if and only if $\mathsf{F}_j(s_0, s_1, \ldots, s_{k_i-1}) = t$ belongs to Δ.

Finally, we interpret the individual constants of $\mathcal{L}'$. For each $i < p$, we define $c_i^{\mathcal{M}} := [\mathsf{c}_i]$. For each i, we define $a_i^{\mathcal{M}} := [\mathsf{a}_i]$. This completes the definition of the structure $\mathcal{M}$.

Now, we wish to prove that that for every term $t = t(x_0, \ldots, x_n)$ for every sequence $s_0, s_1, \ldots, s_n$ of elements of $\mathbf{Term}(\mathcal{L}')$:

$$\mathcal{M} \vDash \varphi\big[[s_0], [s_1], \ldots, [s_n]\big] \text{ if and only if } \varphi(s_0, s_1, \ldots, s_n) \text{ belongs to } \Delta$$

We begin with proving this claim for basic formulas via induction. We will also use the properties (i) to (viii). We will use $t^{\mathcal{M}}$ to denote the equivalence class containing a closed term t.

Lemma 1 For any term $t = t(x_0, \ldots, x_n)$, for any sequence $a_0, \ldots, a_n$ of the elements $\mathbf{Term}(\mathcal{L}')$:

$$t^{\mathcal{M}}\big[[a_0], \ldots, [a_n]\big] = [t(a_0, \ldots, a_n)] \tag{1}$$

Note that this statement is equivalent to $t^{\mathcal{M}}([a_0], \ldots, [a_n]) = [b]$ if and only if $t(a_0, \ldots, a_n) = b$ belongs to Δ.

Proof. We prove this claim by induction on the construction of t. For simplicity, we will write the proof as if t contains at most one free variable. The general case only involves more complicated notation. Also, we have to remind ourselves the definition of a term, which is either an individual variable, an individual constant, or the finite application of functions to finitely many individual variables and constants.

(i) If t is a constant symbol c, then $t^{\mathcal{M}} = c^{\mathcal{M}} = [c]$. By the definition of the equivalence relation $\sim$, $[c] = [d]$ if and only if $c = d$ belongs to Δ.

(ii) t cannot possibly be a variable, as it a closed term in $\mathcal{L}'$.

(iii) We suppose the result has been proven for terms $t_0, \ldots, t_{n-1}$, and we let f be a function symbol. For each i, the sentence $\exists y[t_i(c) = y]$ is a tautology, and therefore is contained in Δ. By construction of Δ, there exists a constant e_i (one of added constants a_k for some arbitrary k) such that $t_i(c) = \mathsf{e}_i$. By induction hypothesis, we have $t^{\mathcal{M}}([c]) = [\mathsf{e}_\mathsf{i}]$.

Thus, we have the following :

$$
\begin{aligned}
\mathsf{F}_j(t_1,\dots,t_n)^{\mathcal{M}}([c]) = [d] \quad &\text{iff} \quad f_j^{\mathcal{M}}(t_1^{\mathcal{M}}[c],\dots,t_n^{\mathcal{M}}[c]) = [d] && \text{(Definition of } L\text{-structures.)}\\
&\text{iff} \quad f_j^{\mathcal{M}}([\mathsf{e}_1],\dots,[\mathsf{e}_n]) = [d] && \text{(Induction Hypothesis)}\\
&\text{iff} \quad \mathsf{F}_j(\mathsf{e}_1,\dots,\mathsf{e}_n) = d \in \Delta && \text{(Definition of } \mathcal{M})\\
&\text{iff} \quad \mathsf{F}_j(t_0(c),\dots,t_i(c)) = d \in \Delta && \text{(By property (vii))}\\
&\text{iff} \quad \big[\mathsf{F}_j(t_0(c),\dots,t_i(c))\big] = [d] && \text{(Definition of } [d_i] \text{)}
\end{aligned}
$$

□

Now that we have proven (1), we now prove it for the relations of $\mathcal{M}$.

Lemma 2 For any formula ϕ with free variables in $\vec{x}$ and constant symbols $c_1,\dots,c_n$,

$$
\mathcal{M} \vDash_{\vec{x}} \phi([c_0],\dots,[c_n]) \quad \text{iff} \quad \phi(c_1,\dots,c_n) \in \Delta \tag{2}
$$

We prove this claim by induction on the construction of ϕ. In some steps, we will write the formula ϕ as if it only has one free variable x; but the proof generalizes directly to the case of any number of free variables.

(i) In the case $\phi(x)$ has the form $t_1(x) = t_2(x)$: Since the sentence, $\exists y[t_i(\mathsf{c}) = y]$ is a tautology, we know it belongs to Δ. Hence, by construction of Δ, there must be a constant d such that $t_i(\mathsf{c}) = \mathsf{d}_i \in \Delta$. By (1), we have that $t_i^{\mathcal{M}}([\mathsf{c}]) = [\mathsf{d_i}]$ for $i = 1, 2$. Thus,

$$
\begin{aligned}
M \vDash \phi(c) \quad &\text{iff} \quad t_1^{\mathcal{M}}([c]) = t_2^{\mathcal{M}}([c]) && \text{(Definition of } L\text{-structures)}\\
&\text{iff} \quad [\mathsf{d}_1] = [\mathsf{d}_2] && (t_i^{\mathcal{M}}([\mathsf{c}]) = [\mathsf{d_i}])\\
&\text{iff} \quad \mathsf{d}_1 \sim_\Delta \mathsf{d}_2 && \text{(Definition of } [\mathsf{d_i}] \text{)}\\
&\text{iff} \quad \mathsf{d}_1 = \mathsf{d}_2 \in \Delta && \text{(Definition of } \sim_\Delta \text{)}\\
&\text{iff} \quad t_1(\mathsf{d}_1) = t_1(\mathsf{d}_2) \in \Delta && (t_i(\mathsf{c}_i) = \mathsf{d}_i)
\end{aligned}
$$

(ii) Suppose that $\phi(x)$ of the form $\mathsf{R}_i(t_1(x),\dots,t_n(x))$. Using the method used earlier, we can find constant symbols $\mathsf{d}_1,\dots,\mathsf{d}_n$ such that $t_i(x) = \mathsf{d}_i$ for $i = 1,\dots,n$. By (1), we have that $t_i^{\mathcal{M}}([\mathsf{c}]) = [\mathsf{d}_i]$. Thus,

$$
\begin{aligned}
\mathcal{M} \vDash \phi(x) \quad &\text{iff} \quad \langle t_1^{\mathcal{M}}([\mathsf{c}]),\dots,t_i^{\mathcal{M}}([\mathsf{c}]\rangle \in R_i^{\mathcal{M}} && \text{(Definition of } L\text{-structures)}\\
&\text{iff} \quad \langle [\mathsf{d}_1],\dots,[\mathsf{d}_i]\rangle \in R_i^{\mathcal{M}} && (\ t_i^{\mathcal{M}}([\mathsf{c}]) = [\mathsf{d}_i]\ )\\
&\text{iff} \quad \mathsf{R}_i(\mathsf{d}_1,\dots,\mathsf{d}_i) \in \Delta && \text{(Definition of } R_i^{\mathcal{M}} \text{)}\\
&\text{iff} \quad \mathsf{R}_i(t_1(\mathsf{c}),\dots,t_i(\mathsf{c})) \in \Delta && \text{(Definition of } R_i^{\mathcal{M}} \text{)}
\end{aligned}
$$

(iii) Suppose $\phi(x)$ is of the form $\varphi(x) \wedge \psi(x)$. By induction hypothesis, we suppose that the claim (2) has been proven for (φ) and (ψ). We show that the claim holds for $\varphi(x) \wedge \psi(x)$:

$$
\begin{aligned}
\mathcal{M} \vDash \varphi(x) \wedge \psi(x) \quad &\text{iff} \quad \mathcal{M} \vDash \varphi(x) \text{ and } \mathcal{M} \vDash \psi(x) && \text{(defn of } L\text{-structures)}\\
&\text{iff} \quad \varphi(x) \in \Delta \text{ and } \psi(x) \in \Delta && \text{(Inductive Hypothesis)}\\
&\text{iff} \quad \varphi(x) \wedge \psi(x) \in \Delta && \text{(By property (v))}
\end{aligned}
$$

(iv) Suppose the claim holds for $\phi(x)$. We prove it holds for $\neg\phi(x)$:

$$\begin{array}{rlll} \mathcal{M} \models \neg\phi(x) & \text{iff} & \mathcal{M} \not\models \phi(x) & (\text{ defn of } L-\text{structures }) \\ & \text{iff} & \phi(x) \notin \Delta & (\text{ Induction Hypothesis }) \\ & \text{iff} & \neg\phi(x) \in \Delta & (\text{ Property (iv) }) \end{array}$$

(v) For the case of the existential quantifier, we consider a formula $\phi(x)$ with one free variable. Supposing the claim has been proven true for $\phi(x)$, we show that it also holds for $\exists x\phi(x)$:

$$\begin{array}{rlll} \mathcal{M} \models \exists x\phi(x) & \text{iff} & \mathcal{M} \models \phi([s]) \text{ for } [s] \in A;\ s \text{ is closed term in } \mathcal{L}' & (\text{ defn of } L-\text{structures }) \\ & \text{iff} & \phi(s) \in \Delta & (\text{ Induction Hypothesis }) \\ & \text{iff} & s = \mathsf{a}_k^{\mathcal{M}} \text{ for some } k \in \mathbb{N} \text{ and } \exists x\phi(x) \in \Delta & (\text{ Construction of } \Delta \text{ }) \end{array}$$

Now we have proven (2), and by property (ii), we have that Γ has as model $\mathcal{M}$. □

References

[1] Hans Halvorson. *Compactness Theorem*. `https://www.princeton.edu/~hhalvors/teaching/phi312_s2013/compactness.pdf`.

[2] David Marker. *Model theory: An Introduction*. Vol. 217. Springer Science & Business Media, 2006.